Towards a fully functorial directed type theory

Fernando Chu

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The idea

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   - Add a hom type constructor, as in North [2].
   - Annotate the variances, $x \overset{\omega}{:} X$ with $\omega \in \{+, -, \circ\}$, as in Nuyts [3].
   - Add a new context extension operation, capturing dependent 2-sided fibrations.

The Grothendieck construction

Definition (Grothendieck construction)

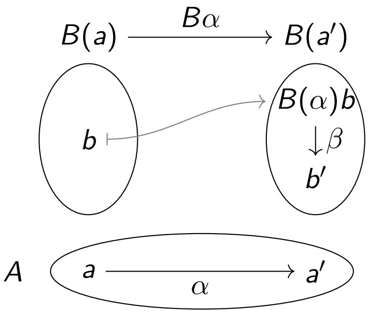
Let A be a category and $B : A \to \mathrm{Cat}$ a functor. The (covariant) **Grothendieck construction** is the category $A.B$ that has:

Objects: Pairs (a, b) with $a : A$ and $b : B(a)$.
Morphisms: A morphism $(a, b) \to (a', b')$ is a pair (α, β) with $\alpha : a \to a'$ and $\beta : B(\alpha)(b) \to b'$.
...

It is the categorification of a dependent sum.

The Grothendieck construction

Graphically, its morphisms look like this:



Note that there is a projection $\pi_A : A.B \to A$, which is in fact an opfibration.

The groupoid model

Hofmann and Streicher's model [1] is as follows:

- Contexts $\rightsquigarrow$ Groupoids
  - Empty context $\rightsquigarrow \star$
- Types in context $\rightsquigarrow$ Functors
  - $(\Gamma \vdash A : \mathcal{U}) \rightsquigarrow (A : \Gamma \to \mathrm{Grpd})$
- Context extension $\rightsquigarrow$ Grothendieck construction
  - $(\Gamma, x : A) \rightsquigarrow (\Gamma.A)$
- Terms in context $\rightsquigarrow$ Sections
  - $(\Gamma \vdash x : A) \rightsquigarrow (\Gamma \to \Gamma.A)$

Hence, we interpret:

$$(\cdot \vdash A : \mathcal{U}) \rightsquigarrow (A : \star \to \mathrm{Grpd}) \rightsquigarrow \text{ a groupoid } A$$

$$(a : A \vdash Fa : B) \rightsquigarrow \text{a section } A \to A.B \rightsquigarrow \text{ a functor } A \to B$$

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The hom-Form rule

$$\frac{\vdash A : \mathcal{U}}{a : A, b : A \vdash \mathsf{Id}_A(a, b) : \mathcal{U}} \ \mathsf{Id}\text{-}\textsc{Form}$$

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This is interpreted as the functor hom : $A.A \to \mathrm{Grpd}$.

$$\begin{array}{ccc} a & \xrightarrow{\cong} & a' \\ \downarrow & & \vdots \\ b & \xrightarrow{\cong} & b' \end{array}$$

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$$\begin{array}{ccc} a & \longleftarrow & a' \\ \downarrow & & \vdots \\ b & \longrightarrow & b' \end{array}$$

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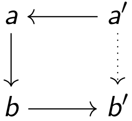

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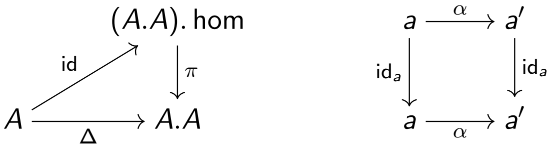
This is interpreted as the morphism id below



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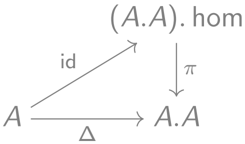

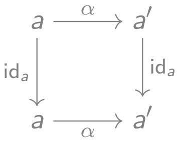

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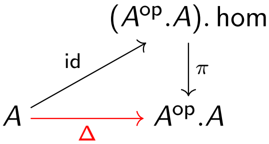

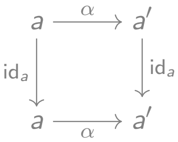

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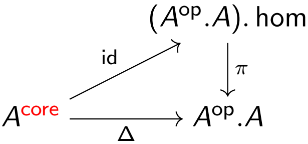

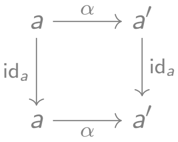

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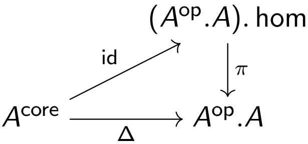

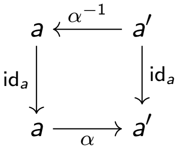

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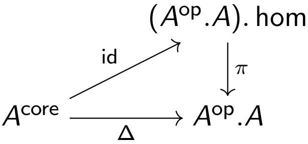

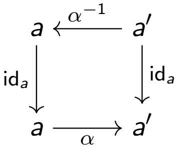

This works! But. . .

Some problems

The core approach works, but it has some problems:

- Restrictive elimination rule.
- Which implies terms are not functorial on all variables, e.g.

$$\frac{a : A \vdash Fa : B}{a : A^{\text{core}}, b : A, f : \hom(a, b) \vdash Ff : \hom(Fa, Fb)}$$

- There obvious translation of a homotopy in HoTT

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  is not natural transformations $F \to G$ in the model.

The ideal hom-Intro rule

The ideal introduction rule would be

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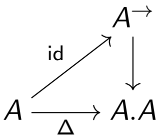
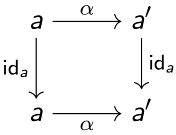
We want this to be interpreted as follows





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We want this to be interpreted as follows

That implies assigning to every profunctor $H : A^{\mathrm{op}} \times A \to \mathrm{Cat}$ an associated functor $\pi : \bar{H} \to A \times A$. The 2-sided fibration [4] associated to H is this construction!

Dependent 2-sided fibrations

Definition (D2SFib)

Let A be a category and $B : A \to \mathrm{Cat}$ a functor. A **dependent 2-sided fibration** (D2SFib) from A to B is a category C equipped with the following data

1. A functor $q : C \to A.B$, together with data specifying that for each $a : A$, the restriction $q_{|a}$ as below

$$\begin{array}{ccccc} C(a) & \xrightarrow{q_{|a}} & (A.B)(a) & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle a} \\ C & \xrightarrow[q]{} & A.B & \xrightarrow[\pi_A]{} & A \end{array}$$

   is a fibration.

Dependent 2-sided fibrations

Definition (D2SFib (cont.))

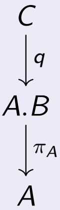
2. Writing $p :\equiv \pi_A \circ q : C \to A$, we require data specifying that p is an opfibration.

Such that

1. q is an opcartesian functor.
2. For each $\alpha : pe \to a$ in A and $\beta : b \to qe$ in $B(p(e))$, the canonical morphism

$$\alpha_! \beta^* e \to (B(\alpha)\beta)^* \alpha_! e$$

given by any of the universal properties is an identity.



Dependent 2-sided fibrations

Proposition

Let A be a category. There is an equivalence of categories

$$\mathsf{Fib}_{split}(A) \simeq \mathsf{Functor}(A^{\mathsf{op}}, \mathsf{Cat})$$

Proposition

Let A and B be categories. There is an equivalence of categories

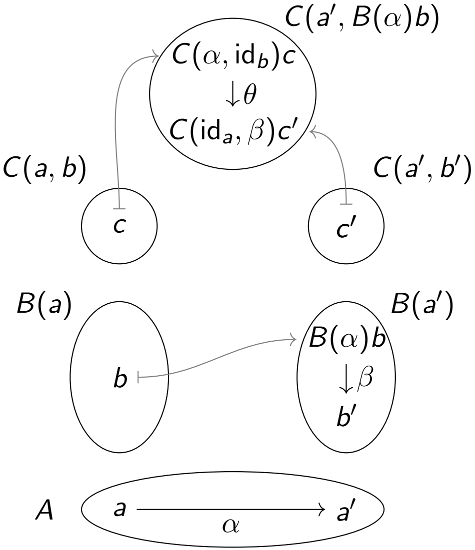
$$\mathsf{2SFib}_{split}(A, B) \simeq \mathsf{Functor}(A \times B^{\mathsf{op}}, \mathsf{Cat})$$

Proposition

Let A be a category and $B : A \to \mathsf{Cat}$ a functor. There is an equivalence of categories

$$\mathsf{D2SFib}_{split}(A, B) \simeq \mathsf{Functor}(A.(\mathsf{op} \circ B), \mathsf{Cat})$$

Dependent 2-sided fibrations



New rules

Coming back to type theory, in addition to the usual context extension rule

$$\frac{\vdash A : \mathcal{U} \qquad a : A \vdash B(a) : \mathcal{U}}{a : A, b \overset{\omega}{:} B(a)\,\mathsf{ctx}}\ \textsc{Ctx-Ext}_1$$

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We get a new context extension operation

$$\frac{\begin{array}{c}\vdash A : \mathcal{U} \qquad a : A \vdash B(a) : \mathcal{U} \\ a : A, b \overset{-}{:} B(a) \vdash C(a,b) : \mathcal{U}\end{array}}{a : A, b : B(a), c \overset{\omega}{:} C(a,b)\,\mathsf{ctx}}\ \textsc{Ctx-Ext}_2$$

New rules

This lets us derive

$$\dfrac{\vdash A : \mathcal{U} \qquad a : A \vdash A : \mathcal{U}}{\dfrac{b : A, a \mathrel{\bar{:}} A \vdash \hom_A(a, b) : \mathcal{U}}{b : A, a : A, f : \hom_A(a, b)\ \mathsf{ctx}} \ \textsc{Ctx-Ext}_2}$$

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Which let us make sense of our introduction rule

$$\dfrac{\vdash A : \mathcal{U}}{a : A \vdash \mathrm{refl}_a : \mathrm{hom}(a, a)}\ \text{hom-}\textsc{Intro}$$

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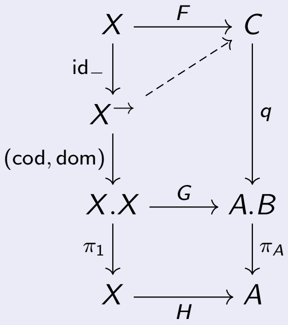
And this also lifts against all opfibrations.

Dependent 2-sided fibrations

Proposition

Let X be a category. If $q : C \to A.B$ is a D2SFib, and we have a commutative diagram as below, with G mapping chosen opcartesian lifts to chosen opcartesian lifts, then there exists a lift as making everything commute.



A new elimination rule

We now obtain a new elimination rule

$$\frac{\vdash A : \mathcal{U} \qquad y : A, x \mathrel{\bar{:}} A \vdash C(x,y) : \mathcal{U} \qquad x : A \vdash c : C(x,x)}{y : A, x : A, f : \hom(x,y) \vdash j_{x,y,f,c} : C(x,y)} \text{ hom-}\textsc{Elim}$$

Which lets us have prove things like:

$$\frac{x : A \vdash Fx : B \qquad y : A, x \mathrel{\bar{:}} A \vdash \hom(Fx, Fy) : \mathcal{U} \qquad x : A \vdash \mathrm{refl}_x : \hom(Fx, Fx)}{y : A, x : A, f : \hom(x,y) \vdash j_{x,y,f,c} : \hom(Fx, Fy)} \text{ hom-}\textsc{Elim}$$

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Some solutions

The D2SFib approach gives some partial solutions:

- Less restrictive elimination rule.
- Terms are fully functorial in all variables:

$$\frac{a : A \vdash Fa : B}{b : A, a : A, f : \hom(a, b) \vdash Ff : \hom(Fa, Fb)}$$

- The obvious translation of a homotopy in HoTT

$$a : A \vdash \varphi_a : \hom(Fa, Ga)$$

  is a natural transformations $F \to G$ in the model.
- We can prove Yoneda inside this theory!

Summary and future work

- We extend the groupoid model of MLTT to a category model.
- We add a hom-type constructor.
- We add a modality to capture contravariance.
- We add a new context extension to capture operations involving the arrow category.
- We hope that these rules (and some more) will allow for reasoning about categories syntactically. We need to strengthen the elimination principle.

Thank you!

References

[1] Martin Hofmann and Thomas Streicher. “The groupoid interpretation of type theory”. In: *Twenty-five years of constructive type theory (Venice, 1995)* 36 (1998), pp. 83–111.

[2] Paige Randall North. “Towards a directed homotopy type theory”. In: *Electronic Notes in Theoretical Computer Science* 347 (2019), pp. 223–239.

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