The groupoid model of MLTT and the category model of Directed Type Theory

Fernando Chu

The Grothendieck construction

Definition

Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathsf{Cat}$ a functor. The *Grothendieck construction* $\int F$ (also $\mathcal{C}.F$) is the category where:

- objects are pairs (Γ, x) with $\Gamma \in \mathcal{C}$ and $x \in F(\Gamma)$;
- a morphism $(\Gamma, x) \to (\Delta, y)$ is a pair of morphisms $f : \Gamma \to \Delta$ in $\mathcal{C}$ and $g : F(f)(x) \to y$ in $F(\Delta)$.

It comes with a forgetful projection $\int F \to \mathcal{C}$.

Compare this to the characterization of identities in Σ-types: for $w, w' : \sum_{x:A} B(x)$,

$$(w = w') \simeq \sum_{p:\mathsf{pr}_1 w = \mathsf{pr}_1 w'} \left(p_*(\mathsf{pr}_2 w) = \mathsf{pr}_2 w'\right).$$

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Categories with Families

Definition

A *category with families* consists of the following.

- A category $\mathcal{C}$.
- A presheaf $\mathsf{Ty} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$.
- A copresheaf $\mathsf{Tm} : \int \mathsf{Ty} \to \mathsf{Set}$ where $\int$ denotes the Grothendieck construction.
- For each object Γ of $\mathcal{C}$ and for each $A \in \mathsf{Ty}(\Gamma)$, there is an object $\pi_\Gamma : \Gamma.A \to \Gamma$ of $\mathcal{C}/\Gamma$ with the following universal property.

$$\hom_{\mathcal{C}/\Gamma}(s : \Delta \to \Gamma, \pi_\Gamma) \cong \mathsf{Tm}(\Delta, \mathsf{Ty}(s)A).$$

Substitution as pullback

Equivalently, the condition

$$\hom_{\mathcal{C}/\Gamma}(s : \Delta \to \Gamma, \pi_\Gamma) \cong \mathsf{Tm}(\Delta, \mathsf{Ty}(s)A).$$

states that for every $s : \Delta \to \Gamma$ and $A \in \mathsf{Ty}(\Gamma)$, the following square is a pullback.

$$\begin{array}{ccc} \Delta.\mathsf{Ty}(s)A & \longrightarrow & \Gamma.A \\ {\scriptstyle \pi_\Delta}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \pi_\Gamma} \\ \Delta & \xrightarrow[s]{} & \Gamma \end{array}$$

The groupoid model

The Hofmann and Streicher 1998 model is as follows:

- Contexts $\rightsquigarrow$ Groupoids
  - Empty context $\rightsquigarrow *$
- Types in context $\rightsquigarrow$ Functors
  - $(\Gamma \vdash A\ \text{Type}) \rightsquigarrow (A : \Gamma \to \text{Grpd})$
- Context extension $\rightsquigarrow$ Grothendieck construction
  - $(\Gamma, x : A) \rightsquigarrow (\Gamma.A)$
- Terms in context $\rightsquigarrow$ Sections
  - $(\Gamma \vdash x : A) \rightsquigarrow (\Gamma \to \Gamma.A)$

Hence, we interpret:

$$(\cdot \vdash A\ \text{Type}) \rightsquigarrow (A : * \to \text{Grpd}) \rightsquigarrow \text{ a groupoid } A$$

$$(a : A \vdash Fa : B) \rightsquigarrow \text{a section } A \to A.B \rightsquigarrow \text{ a functor } A \to B$$

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Recall

Proposition (Straightening-Unstraightening)

Let A be a groupoid. There is an equivalence of categories

$$\mathsf{Isofib}_{\mathit{split}}(A) \simeq \mathsf{Functor}(A, \mathsf{Grpd})$$

Recall that a functor $p : E \to B$ is an *isofibration* if for every object $e \in E$ and every isomorphism $g : p(e) \to b$ in B, we have an isomorphism $\tilde{g} : e \to e'$ in E with $p(\tilde{g}) = g$.

$$e \xrightarrow[\sim]{\tilde{g}} e'$$

$$p(e) \xrightarrow[g]{\sim} b$$

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$$e \overset{\tilde{g}}{\underset{\sim}{\dashrightarrow}} e'$$

$$p(e) \overset{\sim}{\underset{g}{\longrightarrow}} b$$

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Σ-types

Type theory

$$\frac{\Gamma, x : A \vdash B \ \mathsf{Type}}{\Gamma \vdash \Sigma_A B \ \mathsf{Type}} \ \Sigma\text{-}\textsc{Form} \qquad \frac{\Gamma, x : A \vdash B \ \mathsf{Type}}{\Gamma, x : A, y : B \vdash \langle x, y \rangle : \Sigma_A B} \ \Sigma\text{-}\textsc{Intro}$$

Category theory

$$\begin{array}{ccc} \Gamma.A.B & \xrightarrow{\ \Diamond\ } & \Gamma.\Sigma_A B \\ {\scriptstyle \pi_{\Gamma.A}} \downarrow & & \downarrow {\scriptstyle \pi_\Gamma} \\ \Gamma.A & \xrightarrow[\ \pi_\Gamma\ ]{} & \Gamma \end{array}$$

The $\to$-Form rule

Type theory

$$\frac{\vdash A\ \mathsf{Type} \qquad \vdash B\ \mathsf{Type}}{\vdash A \to B\ \mathsf{Type}}\ \to\text{-}\textsc{Form} \qquad\qquad \frac{a : A \vdash b : B}{\vdash f : A \to B}\ \to\text{-}\textsc{Other}$$

Category theory

The objects of $A \to B$ have to correspond to functors $A \to B$.

$$A \to B \ \rightsquigarrow\ \mathsf{Grpd}(A, B).$$

The Id-Form rule

Type theory

$$\frac{\vdash A \ \mathsf{Type}}{a : A, b : A \vdash \mathsf{Id}_A(a, b) \ \mathsf{Type}} \ \mathsf{Id}\text{-}\textsc{Form}$$

Category theory

$$\mathsf{Id}_A : A.A \to \mathsf{Grpd} \qquad \text{or} \qquad \begin{array}{c} A.A.\mathsf{Id}_A \\ \downarrow{\scriptstyle \pi_{\mathsf{Id}_A}} \\ A.A \end{array}$$

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Type theory

$$\frac{\vdash A \ \mathsf{Type}}{a : A, b : A \vdash \mathsf{Id}_A(a,b) \ \mathsf{Type}} \ \mathsf{Id}\text{-}\textsc{Form}$$

Category theory

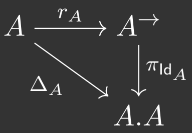
$$\mathsf{Hom}_A : A.A \to \mathsf{Grpd} \qquad \text{or} \qquad A^{\to} \xrightarrow{\pi_{\mathsf{Id}_A}} A.A$$

The Id-Intro rule

Type theory

$$\frac{\vdash A\ \mathsf{Type}}{a : A \vdash \mathsf{refl}_a : \mathsf{Id}_A(a,a)}\ \text{Id-Intro}$$

Category theory



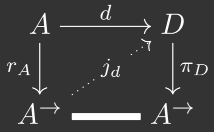
$$r_A : a \mapsto \mathsf{id}_a$$

The Id-Elim rule

Type theory

$$\frac{a : A, b : A,\ p : \mathsf{Id}_A(a,b) \vdash D\ \mathsf{Type} \qquad a : A \vdash d : D[b/a, p/\mathsf{refl}_a]}{a : A, b : A,\ p : \mathsf{Id}_A(a,b) \vdash j_d : D}\ \mathsf{Id}\text{-}\textsc{Elim}$$

Category theory



The Id-Elim rule — proof sketch

Let $f : a \to b$ be an object of $A^{\to}$. Note that $d(a) \in D$ is over $\mathsf{id}_a \in A^{\to}$. Since D is an isofibration:

$$\begin{array}{ccc} a & \xrightarrow{\mathsf{id}_a} & a \\ {\scriptstyle \mathsf{id}_a}\downarrow & & \downarrow{\scriptstyle f} \\ a & \xrightarrow[f]{} & b \end{array}$$

$$d(a) \dashrightarrow j_d(f)$$

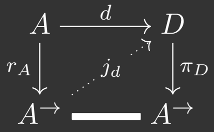
$$(\mathsf{id}_a : a \to a) \xrightarrow[(\mathsf{id}_a, f)]{} (f : a \to b)$$

The Id-Comp rule

Type theory

$$\frac{\begin{array}{c} a : A, b : A,\ p : \mathsf{Id}_A(a, b) \vdash D \ \mathsf{Type} \\ a : A \vdash d : D[b/a, p/\mathsf{refl}_a] \end{array}}{a : A \vdash j_d[a/b, \mathsf{refl}_a/p] \equiv d : D[b/a, p/\mathsf{refl}_a]} \ \mathsf{Id}\text{-}\textsc{Comp}$$

Category theory



Recapping

Briefly

Id-*Form*, Id-*Intro:*
There is a factorization $A \xrightarrow{r_A} \mathsf{Id}_A \xrightarrow{p} A.A$

Id-*Elim*, Id-*Comp:*
The path object $A \xrightarrow{r_A} A^{\rightarrow}$ *lifts against isofibrations.*

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Id-*Form*, Id-*Intro:*
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Universes

Type theory

$$\frac{}{\vdash \mathcal{U}\ \mathsf{Type}}\ \mathcal{U}\text{-}\textsc{Form} \qquad \frac{\vdash A\ \mathsf{Type}}{\vdash A : \mathcal{U}}\ \mathcal{U}\text{-}\textsc{Intro}$$

Category theory

The universe $\mathcal{U}$ is the groupoid of small groupoids and isomorphisms between them.

Equivalences

Definition

An *equivalence* between types A and B is a function equipped with a proof that it is invertible:

$$(A \simeq B) :\equiv \sum_{f:A\to B} \left(\sum_{g:B\to A} g \circ f = \mathsf{id}_A \right) \times \left(\sum_{h:B\to A} f \circ h = \mathsf{id}_B \right).$$

Semantically, $A \simeq B$ is the *groupoid of equivalences* between A and B.

Univalence for sets

The univalence axiom asserts that the canonical map

$$\mathsf{idtoeqv} : (A = B) \xrightarrow{\sim} (A \simeq B)$$

is an equivalence.

This doesn't hold! The left-hand side is isomorphisms, the right-hand side is equivalences. These do coincide when A and B are sets!

Idea: We need -groupoids to get univalence for -groupoids!

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Idea: We need 2-groupoids to get univalence for 1-groupoids!

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Idea: We need $(n+1)$-groupoids to get univalence for n-groupoids!

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Idea: We need ∞-groupoids to get univalence for ∞-groupoids!

Generalizing

HoTT is good for reasoning about:

1. equivalent mathematical structures $A \cong B$;
2. ∞-groupoids (a.k.a. spaces).

Could we generalize?

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1. morphisms of mathematical structures $A \to B$;
2. ∞-categories.

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Could we generalize?

This is the aim of *directed type theory*.

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The category model

The category model is* as follows:

- Contexts $\rightsquigarrow$ Categories
  - Empty context $\rightsquigarrow *$
- Types in context $\rightsquigarrow$ Opfibrations
  - $(\Gamma \vdash A \text{ Type}) \rightsquigarrow (\Gamma.A \to \Gamma)$
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The Id-Form rule

Type theory

$$\frac{\vdash A\ \mathsf{Type}}{a : A, b : A \vdash \mathsf{Id}_A(a,b)\ \mathsf{Type}}\ \mathsf{Id}\text{-}\textsc{Form}$$

Category theory

$$\begin{array}{c} A^{\rightarrow} \\ \downarrow{\scriptstyle p} \\ A.A \end{array}$$

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Category theory

$$A^{\to} \xrightarrow{\;p\;} A.A$$

The Hom-Form rule

Type theory

$$\frac{\vdash A\ \mathsf{Type}}{a : A, b : A \vdash \mathsf{Hom}_A(a, b)\ \mathsf{Type}}\ \mathsf{Hom}\text{-}\textsc{Form}$$

Category theory

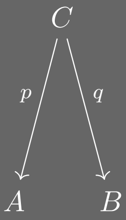
$$\begin{array}{c} A^{\rightarrow} \\ \big\downarrow{\scriptstyle p} \\ A.A \end{array}$$

2-sided fibrations

Definition (2SFib, Street 1974)

Let A : Cat and B : Cat. A **2-Sided Fibration** (2SFib) from A to B is a category C equipped with the following data

1. A span (p, q) from A to B.
2. Evidence that p is an opfibration.
3. Evidence that q is a fibration.
4. Such that some coherences hold.



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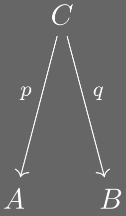

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Let A : Cat and B : Cat. A **2-Sided Fibration** (2SFib) from A to B is a category C equipped with the following data

1. A functor $q : C \to A \times B$.
2. Evidence that $\pi_A \circ q$ is an opfibration.
3. Evidence that $\pi_B \circ q$ is a fibration.
4. Such that some coherences hold.

$$\begin{array}{c} C \\ \downarrow{\scriptstyle q} \\ A \times B \end{array}$$

The Hom-Form rule

Type theory

$$\frac{\vdash A\ \mathsf{Type}}{b : A, a : A \vdash \mathsf{Hom}_A(a, b)\ \mathsf{Type}}\ \mathsf{Hom}\text{-}\textsc{Form}$$

Category theory

$$\begin{array}{c} A^{\to} \\ \big\downarrow \pi \\ A.A \end{array}$$

The Hom-Form rule

Type theory

$$\frac{\vdash A \ \mathsf{Type}}{b : A, a : A \vdash \mathsf{Hom}_A(a, b) \ \mathsf{Type}_2} \ \text{Hom-}\textsc{Form}$$

Category theory

$$\begin{array}{c} A^{\rightarrow} \\ \big\downarrow{\scriptstyle \pi} \\ A.A \end{array}$$

The Hom-Form rule

Type theory

$$\frac{\Gamma \vdash A \text{ Type}}{\Gamma, b : A, a : A \vdash \mathsf{Hom}_A(a, b) \text{ Type}_2} \text{ Hom-Form}$$

Category theory

$$\begin{array}{c} \Gamma.A^{\rightarrow} \\ \downarrow \pi \\ \Gamma.A.A \end{array}$$

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Dependent 2-sided fibrations

Definition (D2SFib)

Let A : Cat and $B : A \to$ Cat. A Dependent **2-Sided Fibration** (D2SFib) from A to B is a category C equipped with the following data

1. A functor $q : C \to A.B$.
2. Evidence that $\pi_A \circ q$ is an opfibration.
3. Evidence that for each $a : A$, the restriction of q to the fiber over a is a fibration.
4. Such that some coherences hold.

$$C \xrightarrow{q} A.B \xrightarrow{\pi_A} A$$

Dependent 2-sided factorization

Factorization on a category

- a factorization of every morphism

$$X \xrightarrow{f} Y \;\mapsto\; X \xrightarrow{\lambda(f)} M(f) \xrightarrow{\rho(f)} Y$$

- that extends to morphisms of morphisms

Dependent 2-sided factorization on a category

- a factorization of every dependent span into a **shoot**

$$C \xrightarrow{q} B \xrightarrow{p} A \;\mapsto\; C \xrightarrow{\lambda(p,q)} M(p,q) \xrightarrow{q} B \xrightarrow{p} A$$

- that extends to morphisms of dependent spans

Path objects

Path objects

We can factorize the diagonal map $X \to X \times X$ as

$$X \xrightarrow{\mathsf{id}_{-}} X^{\to} \xrightarrow{\langle \mathsf{cod}, \mathsf{dom} \rangle} X \times X$$

(Dependent directed) path objects

We can factorize the diagonal dependent span $\Gamma.X \to \Gamma.X.X \xrightarrow{\pi} \Gamma.X$ as

$$\Gamma.X \xrightarrow{\mathsf{id}_{-}} \Gamma.X^{\to} \xrightarrow{\langle \mathsf{cod}, \mathsf{dom} \rangle} \Gamma.X.X \xrightarrow{\pi} \Gamma.X$$

Path objects lift against D2SFibs

Proposition

Let X be a groupoid. The path object of X lifts against all isofibrations.

$$\begin{array}{ccc} X & \xrightarrow{F} & E \\ {\scriptstyle id\_}\downarrow & \overset{j}{\nearrow} & \downarrow{\scriptstyle p} \\ X^{\to} & \xrightarrow[G]{} & B \end{array}$$

Proposition

Let $\Gamma.X \to \Gamma$ be an opfibration. Its path object lifts against all D2SFibs.

$$\begin{array}{ccc} \Gamma.X & \xrightarrow{F} & C \\ {\scriptstyle id\_}\downarrow & \overset{j}{\nearrow} & \\ \Gamma.X^{\to} & & \downarrow{\scriptstyle q} \\ {\scriptstyle \langle \mathrm{cod,dom} \rangle}\downarrow & & \\ \Gamma.X.X & \xrightarrow{G} & B \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Gamma.X & \xrightarrow[H]{} & A \end{array}$$

The Hom-Elim rule

From this, we get a Hom-elimination principle that closely resembles the one for Id-types.

$$\dfrac{\begin{array}{c}\Gamma \vdash A \text{ Type} \\ \Gamma \triangleright b : A \triangleright a : A \blacktriangleright f : \mathsf{Hom}_A(\bar{a}, b) \vdash D \text{ Type} \\ \Gamma \triangleright a : A \vdash d \overset{d}{:} D[\bar{a}/b, \mathsf{refl}_A/f]\end{array}}{\Gamma \triangleright b : A \triangleright a : A \blacktriangleright f : \mathsf{Hom}_A(\bar{a}, b) \vdash j_d : D} \text{ Hom-}\textsc{Elim}$$

Results and future work

- Directed univalence $\mathrm{Hom}(A, B) \xrightarrow{\sim} (A \to B)$ holds for sets.
- One can prove Yoneda lemma synthetically in this framework.
- This lifting property generalizes Street's.
- This framework generalizes North's work on 2-sided factorization systems.
- How do we deal with n-sided fibrations?
- Precisely, when can we strengthen the Hom-elimination rule?

Thank you!

Dependent 2-sided fibrations

Definition (D2SFib)

Let A be a category and $B : A \to \mathsf{Cat}$ a functor. A **dependent 2-sided fibration** (D2SFib) from A to B is a category C equipped with the following data

1. A functor $q : C \to A.B$, together with data specifying that for each $a : A$, the restriction $q_{|a}$ as below

$$\begin{array}{ccccc} C(a) & \xrightarrow{q_{|a}} & (A.B)(a) & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow a \\ C & \xrightarrow[q]{} & A.B & \xrightarrow[\pi_A]{} & A \end{array}$$

   is a fibration.
2. Evidence that $p :\equiv \pi_A \circ q : C \to A$ is an opfibration.

Dependent 2-sided fibrations

Definition (D2SFib (cont.))

Such that

1. q is an opcartesian functor.
2. For each $\alpha : pe \to a$ in A and $\beta : b \to qe$ in $B(p(e))$, the canonical morphism

$$\alpha_! \beta^* e \to (B(\alpha)\beta)^* \alpha_! e$$

given by any of the universal properties is an identity.

$$\begin{array}{c} C \\ \downarrow{\scriptstyle q} \\ A.B \\ \downarrow{\scriptstyle \pi_A} \\ A \end{array}$$

Dependent 2-sided fibrations

Proposition

Let A be a category. There is an equivalence of categories

$$\mathsf{Fib}_{\mathit{split}}(A) \simeq \mathsf{Functor}(A^{\mathsf{op}}, \mathsf{Cat})$$

Proposition

Let A and B be categories. There is an equivalence of categories

$$\mathsf{2SFib}_{\mathit{split}}(A, B) \simeq \mathsf{Functor}(A \times B^{\mathsf{op}}, \mathsf{Cat})$$

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Dependent 2-sided fibrations

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Proposition

Let A be a category and $B : A \to \mathsf{Cat}$ a functor. There is an equivalence of categories

$$\mathsf{D2SFib}_{\mathit{split}}(A, B) \simeq \mathsf{Functor}(A.(\mathsf{op} \circ B), \mathsf{Cat})$$

The straightening operation

Given:

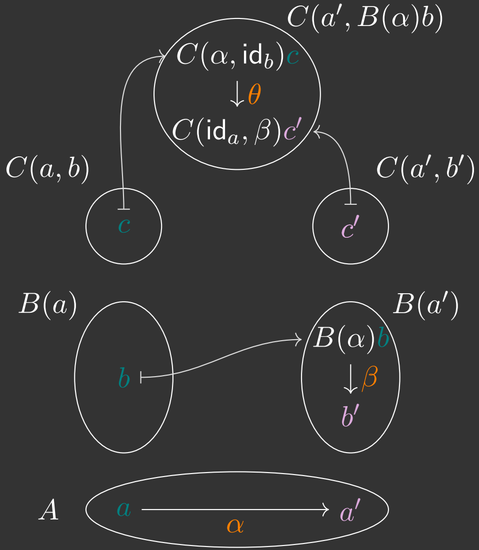
$$A : \mathsf{Cat}$$
$$B : A \to \mathsf{Cat}$$
$$C : A.(\mathsf{op} \circ B) \to \mathsf{Cat}$$

The associated D2SFib is

$$A.\Big(\sum_{\mathsf{op}\circ B} (\mathsf{op} \circ C)\Big)^{\mathsf{op}}$$

We picture a morphism

$$(\alpha, \beta, \theta) : (a, b, c) \to (a', b', c')$$



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