Dependent two-sided fibrations for directed type theory

Fernando Chu & Paige North

Motivation

Extensional Type Theory	$\rightsquigarrow$	Sets
Homotopy Type Theory	$\rightsquigarrow$	Spaces
	$\vdots$	
Directed Type Theory	$\rightsquigarrow$	Categories

The idea

1. We start with MLTT and the groupoid model.
2. Import the rules we see in the semantics back to the syntax, e.g.:
   - Add an op type constructor
   - Add a $\hom$ type constructor
   - Add a new context extension operation, capturing dependent 2-sided fibrations.

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The groupoid model

The Hofmann and Streicher 1998 model is as follows:

- Contexts $\rightsquigarrow$ Groupoids
  - Empty context $\rightsquigarrow \star$
- Types in context $\rightsquigarrow$ Functors
  - $(\Gamma \vdash A : \mathcal{U}) \rightsquigarrow (A : \Gamma \to \mathsf{Grpd})$
- Context extension $\rightsquigarrow$ Grothendieck construction
  - $(\Gamma, x : A) \rightsquigarrow (\Gamma.A)$
- Terms in context $\rightsquigarrow$ Sections
  - $(\Gamma \vdash x : A) \rightsquigarrow (\Gamma \to \Gamma.A)$

Hence, we interpret:

$$(\cdot \vdash A : \mathcal{U}) \rightsquigarrow (A : \star \to \mathsf{Grpd}) \rightsquigarrow \text{ a groupoid } A$$

$$(a : A \vdash Fa : B) \rightsquigarrow \text{a section } A \to A.B \rightsquigarrow \text{ a functor } A \to B$$

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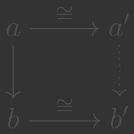
The hom-Form rule

$$\frac{\vdash A : \mathcal{U}}{a : A, b : A \vdash \mathsf{Id}_A(a, b) : \mathcal{U}}\ \mathsf{Id}\text{-}\textsc{Form}$$

This is interpreted as the functor $\mathrm{hom} : A.A \to \mathsf{Grpd}$.



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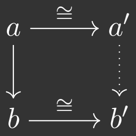

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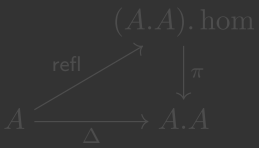

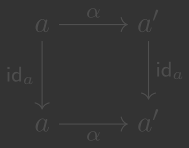

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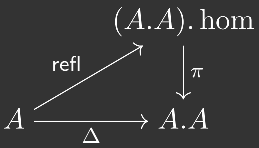

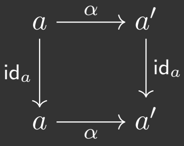

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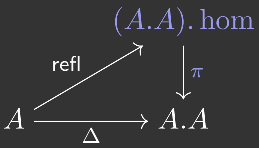

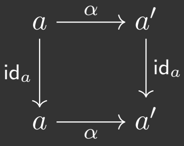

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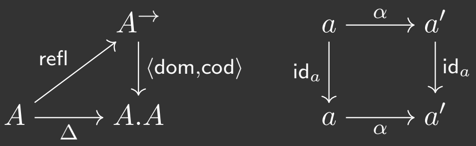

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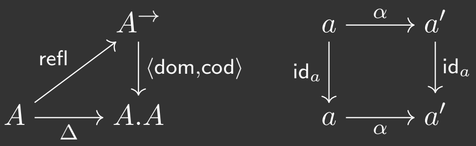

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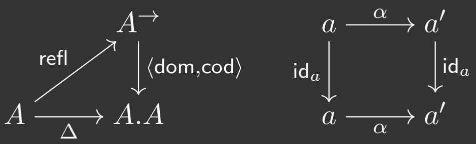
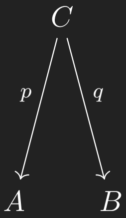


2-sided fibrations

Definition (2SFib, Street 1974)

Let A : Cat and B : Cat. A **2-Sided Fibration** (2SFib) from A to B is a category C equipped with the following data

1. A span (p, q) from A to B.
2. Evidence that p is an opfibration.
3. Evidence that q is a fibration.
4. Such that some coherences hold.



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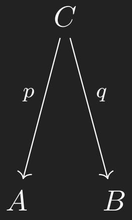

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Definition (2SFib, Street 1974)

Let A : Cat and B : Cat. A **2-Sided Fibration** (2SFib) from A to B is a category C equipped with the following data

1. A functor $q : C \to A \times B$.
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$$C \xrightarrow{q} A \times B \xrightarrow{\pi_A} A$$

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Dependent 2-sided fibrations

Definition (D2SFib)

Let A : Cat and $B : A \to \text{Cat}$. A Dependent **2-Sided Fibration** (D2SFib) from A to B is a category C equipped with the following data

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Dependent 2-sided fibrations

Proposition

Let A be a category. There is an equivalence of categories

$$\mathsf{Fib}_{split}(A) \simeq \mathsf{Functor}(A^{\mathsf{op}}, \mathsf{Cat})$$

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$$\mathsf{D2SFib}_{\mathit{split}}(A, B) \simeq \mathsf{Functor}(A.(\mathsf{op} \circ B), \mathsf{Cat})$$

A new context extension

In addition to

$$\frac{\vdash A : \mathcal{U} \qquad a : A \vdash B(a) : \mathcal{U}}{a : A, b : B(a)\ \mathsf{ctx}}\ \textsc{Ctx-Ext}_1$$

We now add

$$\frac{\vdash A : \mathcal{U} \qquad a : A \vdash B(a) : \mathcal{U} \qquad a : A, b : B(a)^{\mathsf{op}} \vdash C(a,b) : \mathcal{U}}{a : A, b : B(a), c \overset{2\mathfrak{f}}{:} C(a,b)\ \mathsf{ctx}}\ \textsc{Ctx-Ext}_2$$

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A new hom-intro rule

This lets us derive

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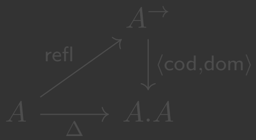
Which let us make sense of our introduction rule

$$\dfrac{\vdash A : \mathcal{U}}{a : A \vdash \mathsf{refl}_a \overset{\partial}{:} \hom(a,a)}\ \hom\text{-}\textsc{Intro}$$



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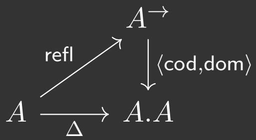

A new hom-elim rule

We now obtain a new elimination rule

$$\frac{\begin{array}{c}\Gamma, b : A, a : A, f \overset{2\mathfrak{f}}{:} \hom_A(a,b) \vdash D : \mathcal{U} \\ \Gamma, a : A \vdash d : D[a/b, \mathsf{refl}_A/f]\end{array}}{\Gamma, b : A, a : A, f \overset{2\mathfrak{f}}{:} \hom_A(a,b) \vdash j_d : D} \ \hom\text{-}\textsc{Elim}$$

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$$\dfrac{\begin{array}{c}\Gamma, a : A \vdash X : \mathcal{U} \\ \Gamma, b : A, a : A, f \overset{2\mathfrak{f}}{:} \hom_A(a,b), x : X^{\mathsf{op}} \vdash D : \mathcal{U} \\ \Gamma, a : A, x : X \vdash d \overset{\mathfrak{d}}{:} D[a/b, \mathsf{refl}_A/f]\end{array}}{\Gamma, b : A, a : A, f \overset{2\mathfrak{f}}{:} \hom_A(a,b), x : X \vdash j_d \overset{2\mathfrak{f}}{:} D} \text{ hom-}\textsc{Elim}$$

Some solutions

The D2SFib approach gives some partial solutions:

- Terms are fully functorial in all variables:

$$\frac{a : A \vdash Fa : B}{b : A, a : A, f \overset{2\mathfrak{f}}{:} \hom(a,b) \vdash Ff : \hom(Fa, Fb)}$$

- The analog of a homotopy in HoTT

$$a : A \vdash \varphi_a \overset{\mathfrak{d}}{:} \hom(Fa, Ga)$$

  is interpreted as a natural transformation $F \to G$ in the model.

- We can prove Yoneda inside this theory!

Summary

We start form the groupoid model and add:

- Categories as types.
- A $\hom$-type constructor.
- The op type constructor.
- A new context extension, which recovers the arrow category.

Future work

- Better understanding of D2SFibs
  - (D2S) factorization systems?
  - Stability under pullback?
  - How do they interact with Π-types?
  - Characterization as a lax normal functor $A.B \to$ Prof?
  - Dependent n-sided fibrations?
- Remove of explicit substitutions?
- How to write a typechecker for this?

Thank you!

The straightening operation

Given:

$A : \mathsf{Cat}$

$B : A \to \mathsf{Cat}$

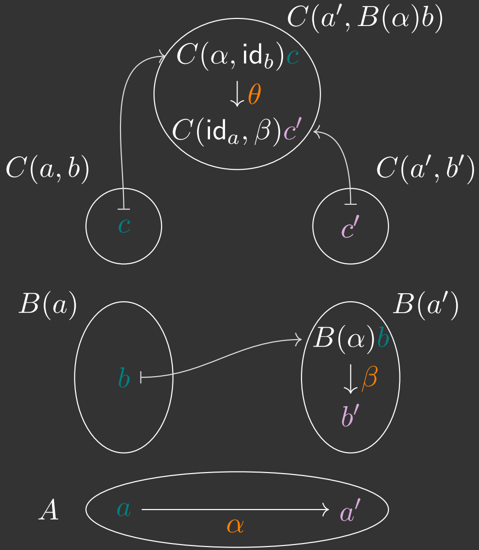
$C : A.(\mathsf{op} \circ B) \to \mathsf{Cat}$

The associated D2SFib is

$$A.\Big(\sum_{\mathsf{op} \circ B} (\mathsf{op} \circ C) \Big)^{\mathsf{op}}$$

We picture a morphism

$(\alpha, \beta, \theta) : (a, b, c) \to (a', b', c')$



Dependent 2-sided fibrations

Definition (D2SFib)

Let A be a category and $B : A \to \mathrm{Cat}$ a functor. A **dependent 2-sided fibration** (D2SFib) from A to B is a category C equipped with the following data

1. A functor $q : C \to A.B$, together with data specifying that for each $a : A$, the restriction $q_{|a}$ as below

$$\begin{array}{ccccc} C(a) & \xrightarrow{q_{|a}} & (A.B)(a) & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow a \\ C & \xrightarrow[q]{} & A.B & \xrightarrow[\pi_A]{} & A \end{array}$$

   is a fibration.

2. Evidence that $p :\equiv \pi_A \circ q : C \to A$ is an opfibration.

Dependent 2-sided fibrations

Definition (D2SFib (cont.))

Such that

1. q is an opcartesian functor.
2. For each $\alpha : pe \to a$ in A and $\beta : b \to qe$ in $B(p(e))$, the canonical morphism

   $$\alpha_! \beta^* e \to (B(\alpha)\beta)^* \alpha_! e$$

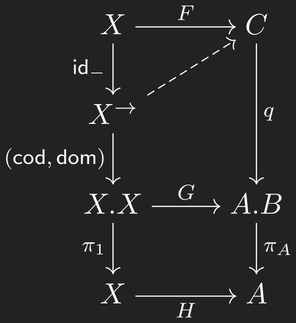
   given by any of the universal properties is an identity.

$$\begin{array}{c} C \\ \big\downarrow{\scriptstyle q} \\ A.B \\ \big\downarrow{\scriptstyle \pi_A} \\ A \end{array}$$

A lifting property

Proposition

Let X be a category. If $q : C \to A.B$ is a D2SFib, and we have a commutative diagram as below, with G mapping chosen opcartesian lifts to chosen opcartesian lifts, then there exists a lift as making everything commute.



References

- Hofmann, Martin and Thomas Streicher (1998). "The groupoid interpretation of type theory". In: *Twenty-five years of constructive type theory (Venice, 1995)* 36, pp. 83–111.
- Street, Ross (1974). "Fibrations and Yoneda's lemma in a 2-category". In: *Category Seminar*. Ed. by Gregory M. Kelly. Vol. 420. Series Title: Lecture Notes in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 104–133. ISBN: 978-3-540-06966-9 978-3-540-37270-7. DOI: 10.1007/BFb0063102.