Dependent-two sided factorization systems for directed type theory

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Background

In recent years, Homotopy Type Theory (HoTT) has had great success both as a foundation of mathematics and as internal language to reason about ∞-groupoids (a.k.a spaces). However, in many areas of mathematics and computer science, it is often the case that it is categories, not groupoids, which are the more important structures to consider. For this reason, multiple Directed Type Theories have been proposed [1, 2, 3, 4, 5, 6]; i.e. theories whose semantics are based on categories.

In previous work presented at TYPES 2025 [7], we introduced such a type theory, Twisted Type Theory (TTT). It featured a novel "twisting" operation on types: given a type that depends both contravariantly and covariantly on some variables, its twist is a new type that depends only covariantly on the same variables. To provide the semantics of this operation, we introduced the notion of dependent 2-sided fibrations (D2SFibs), which generalize Street's notion of 2-sided fibrations [8] and sketched the basic theory of these, as well as characterize them through a straightening-unstraightening theorem. These results allowed for a novel elimination rule for Hom-types, which allows for the development of some synthetic category theory.

While we argued that TTT is a natural extension of Hofmann and Streicher's groupoid model [9] to the directed case, some of our construction were ad-hoc; for example, the Hom-elimination rule. In this abstract, we present some work in progress to remedy this situation, via an exploration of a factorization systems tailored to the study of D2SFibs.

Dependent 2-sided fibrations and Directed Type Theory

In the following, we denote the Grothendieck construction of B as $A \rhd B$. We also make use of Street's definition of an opfibration in a 2-category [8]. For simplicity, all our notions of fibrations will be split. We now recall the definition of D2SFibs and their use in type theory.

Definition 1. Let $\mathcal{C}$ be a 2-category and $p : B \to A$ be an opfibration in $\mathcal{C}$. A *dependent 2-sided fibration* (D2SFib) from A to p is a fibration over p in the category $\mathsf{Opfib}_{\mathcal{C}}(A)$ of opfibrations over A. The category of D2SFibs from A to p is denoted $\mathsf{D2SFib}_{\mathcal{C}}(A, p)$. Whenever p can be inferred from context, we also write $\mathsf{D2SFib}_{\mathcal{C}}(A, B)$ for this same category, and call their elements D2SFibs from A to B.

By unfolding definitions, one can check that for the 2-category of categories Cat a D2SFib from A to B is then a functor $q : C \to B$ such that it is a "local fibration" over B and such that $\pi \circ q : C \to A$ is an opfibration, satisfying certain coherences. As other notions of fibrations, D2SFibs arise as the "unstraightening" of a category of functors.

Proposition 1. *Let A be a category and $B : A \to \mathsf{Cat}$ a functor. There is an equivalence of categories*

$$\phi : [A \rhd (\mathsf{op} \circ B), \mathsf{Cat}] \simeq \mathsf{D2SFib}_{\mathsf{Cat}}(A, A \rhd B)$$

It is this proposition which validates the following rule in the category model [2] of MLTT.

$$\frac{\Gamma\ \mathsf{ctx} \qquad \Gamma \vdash A\ \mathsf{type} \qquad \Gamma, a : A^{\mathsf{op}} \vdash B(a)\ \mathsf{type}}{\Gamma, a : A, b \overset{2\mathsf{f}}{:} B(\bar{a})\ \mathsf{ctx}}\ \textsc{Twist}$$

Indeed, we define the interpretation of the context $\Gamma, a : A, b \overset{2\mathsf{f}}{:} B(\bar{a})$ to be the application of the functor ϕ above to the functor $B : \Gamma \rhd (\mathsf{op} \circ A) \to \mathsf{Cat}$. This twist rule is one of the key ingredients to our development of TTT in forthcoming work.

Dependent 2-sided Factorization Systems

Arguably, one of the most insightful and natural ways in which one can obtain the groupoid model of MLTT is through the Algebraic Weak Factorization System on the category of groupoids [10]. Therefore, we seek an analogous result for categories and D2SFibs.

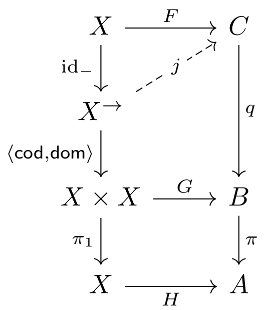
Here, we state some preliminary definitions and results in this direction. We follow closely North's [11] approach to generalizing Weak Factorization Systems (WFSs) to 2-sided fibrations.

Definition 2. A *dependent span* is a pair of two composable morphisms $C \xrightarrow{q} B \xrightarrow{p} A$, with p an opfibration. A morphism between two dependent spans $C \xrightarrow{q} B \xrightarrow{p} A$ and $C' \xrightarrow{q'} B' \xrightarrow{p'} A'$ is a triple $F : C \to C'$, $G : B \to B'$ and $H : C \to C'$ such that $q' \circ F = G \circ q$, and $(G, H) : p \to p'$ is an opcartesian functor.

Definition 3. A *dependent 2-sided factorization system* is a functorial factorization of every dependent span $C \xrightarrow{q} B \xrightarrow{p} A$ into a *shoot*, that is, into a sequence $C \xrightarrow{\lambda(p,q)} M(p,q) \xrightarrow{q} B \xrightarrow{p} A$.

With these two definitions we can now consider fibrations and cofibrations, as well as path objects in any 2-category that has a notion of opfibration. In the case of Cat, we show that the path object has an appropriate lifting property, reminiscent of the lifting property of the path objects in a WFS.

Proposition 2. *Let X be a category. The path object of X lifts against all D2SFibs. That is, for any D2SFib $q : C \to B$ from A to B, and for any morphism of dependent spans as in the diagram on the right, there is a functor $j : X^{\to} \to C$ making the diagram commute.*



Discussion and future work

As was mentioned in the introduction, this is work in progress. Critically, unlike the concepts defined in the previous section, there is no obvious way to adapt North's definition of a 2-sided WFS to the dependent case. Furthermore, the lifting property stated in Proposition 2 is not sufficient to validate to validate the Hom-elimination rule of TTT. Indeed, to do that it would be necessary that $X^{\to} \xrightarrow{\langle \mathsf{cod},\mathsf{dom}\rangle} X \times X \xrightarrow{G} X^{\to}$ is the identity, but no such G is available in general. This suggests that our current approach is not capturing all the structure necessary for obtaining a WFS-like structure that induces our semantics for TTT.

Another important aspect to investigate is the conditions of strictness and splitness in the notions of fibrations we are considering. Without them, the amount of coherences explodes, making them unmanageable. For example, the proposition analogous to Proposition 1 for non-split Street fibrations would be a 2-fibered biequivalence of bicategories, i.e., an isomorphism of pseudofunctors into the tricategory of bicategories. We hope to find mechanism to reduce such statements to simpler ones, involving only reasoning inside a bicategory.

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