Dependent 2-sided factorization systems for directed type theory

Fernando Chu & Paige North

Motivation

Extensional Type Theory	$\rightsquigarrow$	Sets
Homotopy Type Theory	$\rightsquigarrow$	Spaces
	$\vdots$	
Directed Type Theory	$\rightsquigarrow$	Categories

The groupoid model

The Hofmann and Streicher 1998 model is as follows:

- Contexts $\rightsquigarrow$ Groupoids
  - Empty context $\rightsquigarrow *$
- Types in context $\rightsquigarrow$ Functors
  - $(\Gamma \vdash A\,\text{Type}) \rightsquigarrow$ $(A : \Gamma \to \text{Grpd})$
- Context extension $\rightsquigarrow$ Grothendieck construction
  - $(\Gamma, x : A) \rightsquigarrow (\Gamma.A)$
- Terms in context $\rightsquigarrow$ Sections
  - $(\Gamma \vdash x : A) \rightsquigarrow (\Gamma \to \Gamma.A)$

Hence, we interpret:

$$(\cdot \vdash A\,\text{Type}) \rightsquigarrow (A : * \to \text{Grpd}) \rightsquigarrow \text{ a groupoid } A$$

$$(a : A \vdash Fa : B) \rightsquigarrow \text{a section } A \to A.B \rightsquigarrow \text{ a functor } A \to B$$

Recall

Proposition (Straightening-Unstraightening)

Let A be a groupoid. There is an equivalence of categories

$$\mathsf{Isofib}_{\mathit{split}}(A) \simeq \mathsf{Functor}(A, \mathsf{Grpd})$$

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The Id-Form rule

Type theory

$$\frac{\vdash A\ \mathsf{Type}}{a : A, b : A \vdash \mathsf{Id}_A(a, b)\ \mathsf{Type}}\ \mathsf{Id}\text{-}\textsc{Form}$$

Category theory

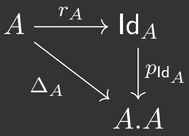
$$\begin{array}{c} \mathsf{Id}_A \\ \big\downarrow {\scriptstyle p_{\mathsf{Id}_A}} \\ A.A \end{array}$$

The Id-Intro rule

Type theory

$$\frac{\vdash A\ \mathsf{Type}}{a : A \vdash \mathsf{refl}_a : \mathsf{Id}_A(a, a)}\ \mathsf{Id}\text{-}\textsc{Intro}$$

Category theory



The Id-Elim rule

Type theory

$$\frac{\begin{array}{c} a : A, b : A,\ p : \mathsf{Id}_A(a,b) \vdash D \ \mathsf{Type} \\ a : A \vdash d : D[b/a, p/\mathsf{refl}_a] \end{array}}{a : A, b : A,\ p : \mathsf{Id}_A(a,b) \vdash j_d : D}\ \mathsf{Id}\text{-}\textsc{Elim}$$

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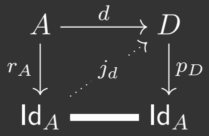

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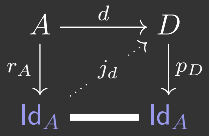

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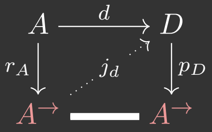

Recapping

Briefly

Id-*Form*, Id-*Intro:*
There is a factorization $A \xrightarrow{r_A} \mathsf{Id}_A \xrightarrow{p} A.A$

Id-*Elim*, Id-*Comp:*
The path object $A \xrightarrow{r_A} A.A$ *lifts against isofibrations.*

What is the directed analogue?

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The category model

The category model is* as follows:

- Contexts $\rightsquigarrow$ Categories
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$$\begin{array}{c} A^{\rightarrow} \\ \big\downarrow{\scriptstyle p} \\ A.A \end{array}$$

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The Hom-Form rule

Type theory

$$\frac{\vdash A\ \mathsf{Type}}{a : A, b : A \vdash \mathsf{Hom}_A(a, b)\ \mathsf{Type}}\ \mathsf{Hom}\text{-}\textsc{Form}$$

Category theory

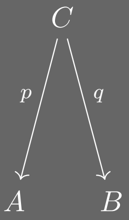
$$\begin{array}{c} A^{\rightarrow} \\ \big\downarrow{\scriptstyle p} \\ A.A \end{array}$$

2-sided fibrations

Definition (2SFib, Street 1974)

Let A : Cat and B : Cat. A **2-Sided Fibration** (2SFib) from A to B is a category C equipped with the following data

1. A span (p, q) from A to B.
2. Evidence that p is an opfibration.
3. Evidence that q is a fibration.
4. Such that some coherences hold.



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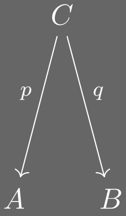

2-sided fibrations

Definition (2SFib, Street 1974)

Let A : Cat and B : Cat. A **2-Sided Fibration** (2SFib) from A to B is a category C equipped with the following data

1. A functor $q : C \to A \times B$.
2. Evidence that $\pi_A \circ q$ is an opfibration.
3. Evidence that $\pi_B \circ q$ is a fibration.
4. Such that some coherences hold.

$$\begin{array}{c} C \\ \downarrow{\scriptstyle q} \\ A \times B \end{array}$$

The Hom-Form rule

Type theory

$$\frac{\vdash A \ \mathsf{Type}}{b : A, a : A \vdash \mathsf{Hom}_A(a, b)\ \mathsf{Type}}\ \mathsf{Hom}\text{-}\textsc{Form}$$

Category theory

$$\begin{array}{c} A^{\rightarrow} \\ \big\downarrow p \\ A.A \end{array}$$

The Hom-Form rule

Type theory

$$\frac{\vdash A \ \mathsf{Type}}{b : A, a : A \vdash \mathsf{Hom}_A(a, b) \ \mathsf{Type}_2} \ \mathsf{Hom}\text{-}\textsc{Form}$$

Category theory

$$\begin{array}{c} A^{\to} \\ \downarrow p \\ A.A \end{array}$$

The Hom-Form rule

Type theory

$$\frac{\Gamma \vdash A \ \mathsf{Type}}{\Gamma, b : A, a : A \vdash \mathsf{Hom}_A(a, b) \ \mathsf{Type}_2} \ \mathsf{Hom}\text{-}\textsc{Form}$$

Category theory

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4. Such that some coherences hold.

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Dependent 2-sided fibrations

Definition (D2SFib)

Let A : Cat and $B : A \to \text{Cat}$. A Dependent **2-Sided Fibration** (D2SFib) from A to B is a category C equipped with the following data

1. A functor $q : C \to A.B$.
2. Evidence that $\pi_A \circ q$ is an opfibration.
3. Evidence that for each $a : A$, the restriction of q to the fiber over a is a fibration.
4. Such that some coherences hold.

Dependent 2-sided factorization

Factorization on a category

- a factorization of every morphism

$$X \xrightarrow{f} Y \;\mapsto\; X \xrightarrow{\lambda(f)} M(f) \xrightarrow{\rho(f)} Y$$

- that extends to morphisms of morphisms

Dependent 2-sided factorization on a category

- a factorization of every dependent span into a **shoot**

$$C \xrightarrow{q} B \xrightarrow{p} A \;\mapsto\; C \xrightarrow{\lambda(p,q)} M(p,q) \xrightarrow{q} B \xrightarrow{p} A$$

- that extends to morphisms of dependent spans

Path objects

Path objects

We can factorize the diagonal map $X \to X \times X$ as

$$X \xrightarrow{\mathsf{id}_{-}} X^{\to} \xrightarrow{\langle \mathsf{cod},\mathsf{dom} \rangle} X \times X$$

(Dependent directed) path objects

We can factorize the diagonal dependent span $\Gamma.X \to \Gamma.X.X \xrightarrow{\pi} \Gamma.X$ as

$$\Gamma.X \xrightarrow{\mathsf{id}_{-}} \Gamma.X^{\to} \xrightarrow{\langle \mathsf{cod},\mathsf{dom} \rangle} \Gamma.X.X \xrightarrow{\pi} \Gamma.X$$

Path objects lift against D2SFibs

Proposition

Let X be a groupoid. The path object of X lifts against all isofibrations.

$$\begin{array}{ccc} X & \xrightarrow{F} & E \\ {\scriptstyle id_{-}}\downarrow & \overset{j}{\nearrow} & \downarrow{\scriptstyle p} \\ X^{\rightarrow} & \xrightarrow[G]{} & B \end{array}$$

Proposition

Let $\Gamma.X \to \Gamma$ be an opfibration. Its path object lifts against all D2SFibs.

$$\begin{array}{ccc} \Gamma.X & \xrightarrow{F} & C \\ {\scriptstyle id_{-}}\downarrow & \overset{j}{\nearrow} & \\ \Gamma.X^{\rightarrow} & & \downarrow{\scriptstyle q} \\ {\scriptstyle \langle \text{cod,dom} \rangle}\downarrow & & \\ \Gamma.X.X & \xrightarrow{G} & B \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Gamma.X & \xrightarrow[H]{} & A \end{array}$$

Some results

- We get a Hom-elimination rule out of this.
- The lifting property is well-behaved (e.g. stable under appropriate pullback)
- This lifting property generalizes Street's.
- One can extend many known notions into this framework.
- This framework generalizes North's work on 2-sided factorization systems.

Future work

- This does not subsume the Hom-elim from the previous TYPES talk.
- Remove splitness of the fibrations involved.
- How to do this in an arbitrary 2-category?
- Can we iterate? n-sided fibrations?

Thank you!

Dependent 2-sided fibrations

Definition (D2SFib)

Let A be a category and $B : A \to \mathsf{Cat}$ a functor. A **dependent 2-sided fibration** (D2SFib) from A to B is a category C equipped with the following data

1. A functor $q : C \to A.B$, together with data specifying that for each $a : A$, the restriction $q_{|a}$ as below

$$\begin{array}{ccccc} C(a) & \xrightarrow{q_{|a}} & (A.B)(a) & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle a} \\ C & \xrightarrow[q]{} & A.B & \xrightarrow[\pi_A]{} & A \end{array}$$

   is a fibration.

2. Evidence that $p :\equiv \pi_A \circ q : C \to A$ is an opfibration.

Dependent 2-sided fibrations

Definition (D2SFib (cont.))

Such that

1. q is an opcartesian functor.
2. For each $\alpha : pe \to a$ in A and $\beta : b \to qe$ in $B(p(e))$, the canonical morphism

$$\alpha_! \beta^* e \to (B(\alpha)\beta)^* \alpha_! e$$

   given by any of the universal properties is an identity.

$$\begin{array}{c} C \\ \downarrow{\scriptstyle q} \\ A.B \\ \downarrow{\scriptstyle \pi_A} \\ A \end{array}$$

Dependent 2-sided fibrations

Proposition

Let A be a category. There is an equivalence of categories

$$\mathsf{Fib}_{\mathit{split}}(A) \simeq \mathsf{Functor}(A^{\mathsf{op}}, \mathsf{Cat})$$

Proposition

Let A and B be categories. There is an equivalence of categories

$$\mathsf{2SFib}_{\mathit{split}}(A, B) \simeq \mathsf{Functor}(A \times B^{\mathsf{op}}, \mathsf{Cat})$$

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Proposition

Let A be a category and $B : A \to \mathsf{Cat}$ a functor. There is an equivalence of categories

$$\mathsf{D2SFib}_{\mathit{split}}(A, B) \simeq \mathsf{Functor}(A.(\mathsf{op} \circ B), \mathsf{Cat})$$

The straightening operation

Given:

$A : \mathsf{Cat}$

$B : A \to \mathsf{Cat}$

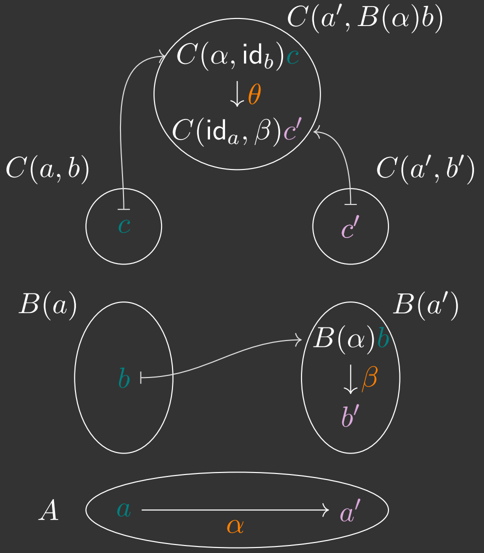
$C : A.(\mathsf{op} \circ B) \to \mathsf{Cat}$

The associated D2SFib is

$$A.\Big(\sum_{\mathsf{op} \circ B} (\mathsf{op} \circ C)\Big)^{\mathsf{op}}$$

We picture a morphism

$(\alpha, \beta, \theta) : (a, b, c) \to (a', b', c')$



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Hofmann, Martin and Thomas Streicher (1998). "The groupoid interpretation of type theory". In: *Twenty-five years of constructive type theory (Venice, 1995)* 36, pp. 83–111.

Street, Ross (1974). "Fibrations and Yoneda's lemma in a 2-category". In: *Category Seminar*. Ed. by Gregory M. Kelly. Vol. 420. Series Title: Lecture Notes in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 104–133. ISBN: 978-3-540-06966-9 978-3-540-37270-7. DOI: 10.1007/BFb0063102.