A directed type theory for 1-categories

Fernando Chu[1], Éléonore Mangel[2], and Paige Randall North[1]

[1] Utrecht University, Netherlands
[2] École normale supérieure Paris-Saclay, France

Background

In Martin-Löf Type Theory (MLTT), the identity types are defined as the family of types $\mathsf{Id}_A(x, y)$ for A a type, and x and y two elements of A, generated inductively by $\mathsf{refl}_a : \mathsf{Id}_A(a, a)$. We can deduce the usual properties of equality: reflexivity, symmetry and transitivity. In fact, Hofmann and Streicher [1] discovered that these properties correspond respectively to the groupoidal operations of identity, inverse and composition, when identities between two terms of a type A are interpreted as morphisms between two objects of A. This discovery showed that MLTT can be used to prove and verify theorems about groupoids and even ∞-groupoids.

Currently, there are many attempts towards finding a similar type theory that instead codifies the structures of (higher) category theory and directed homotopy theory, but no clear consensus has been reached yet. One natural idea to solve this problem is to replace the symmetric identity types with directed homomorphism types. Indeed, whereas groupoid theory and homotopy theory are used to study structures with symmetric paths, (higher) category theory and directed homotopy theory are used to study structures with directed paths.

Related Works

Our work builds on the previous work of the third-listed author [2]. We keep the same goal of having a homomorphism type former with simple rules analogous to the identity type former of the MLTT. We solve one of the main problems of the previous article: our type theory includes both directed homomorphism types and Martin-Löf's original identity types without the collapse of the former into the latter.

We are also inspired by Nuyts [3], especially how variances of assumptions and terms are marked in judgments, and we improve on it by building an interpretation of our syntax.

Our work diverges from other attempts in the literature. Indeed, the works of Licata and Harper [4] and Ahrens, North and Van Der Weide [5] don't have a homomorphism type former and the work of Riehl and Shulman [6] builds on Cubical Type Theory rather than MLTT. Finally, we differ from Kavvos [7] by giving a syntax for our semantics.

Our new attempt

As mentioned, our theory builds upon previous theories by adding orientations $+$, $-$ and $\circ$, while also introducing a hom-type former and an Id-type former. We present a sketch of this construction on the following paragraphs.

We start with MLTT and add orientations that mark the variance of assumptions and terms.

$$\Gamma \vdash_{\ell \vdash \omega} t : A,$$

where ℓ is a list of orientations (one for each type in Γ) and ω is the orientation of the term t.

The first two orientations are $+$ and $-$, corresponding respectively to covariance and contravariance. For example, given a contravariant term $t : B$ depending covariantly on a variable $x : A$ and a morphism $\varphi : a \to a'$ in A, then we obtain a morphism from $t(a)$ to $t(a')$ in B.

However, many mathematical notions that we want to express in our system are neither covariant nor contravariant. For example, the identity type of x and y can depend neither covariantly nor contravariantly on x, nor on y. If it did, it would allow us to transport identity along morphisms and thus the two ends of every morphism would be identified. We want identities to transport along isomorphisms, which motivates the introduction of a third orientation for this case, denoted $\circ$, which will correspond to isovariance.

With $t : B$ a covariant term depending isovariantly on a variable $x : A$, a morphism $\phi : a \to a'$ in A will give us no information between $t(a)$ and $t(a')$. But if ϕ is an isomorphism, then we will have an isomorphism between $t(a)$ and $t(a')$ in B. Conversely, if $t : B$ is an isovariant term depending on $x : A$ covariantly, any morphism from a to a' in A will induce an isomorphism between $t(a)$ and $t(a')$ in B.

We then introduce a $\mathsf{hom}_A(x, y)$ type (following [2]) that respects these orientations in a coherent way. Its first three rules are the following (to which must be added hom-LEFT-ELIM and corresponding computation rules).

$$\textsf{hom-}\textsc{form}\quad \frac{\Gamma \vdash_{\ell \vdash +} A : \mathcal{U}_k}{\Gamma, x : A, y : A \vdash_{\ell, x^-, y \vdash +} \mathsf{hom}_A(x, y) : \mathcal{U}_k} \qquad \textsf{hom-}\textsc{intro}\quad \frac{\Gamma \vdash_{\ell \vdash +} A : \mathcal{U}_k}{\Gamma, x : A \vdash_{\ell, x^\circ \vdash \circ} 1_x : \mathsf{hom}_A(x, x)}$$

$$\textsf{hom-}\textsc{left-elim}\quad \frac{\begin{array}{c}\Gamma \vdash_{\ell \vdash +} A : \mathcal{U}_k \\ \Gamma, x : A, y : A, z : \mathsf{hom}_A(x, y) \vdash_{\ell, x^\circ, y, z \vdash +} D(x, y, z) : \mathcal{U}_k \qquad \Gamma, x : A \vdash_{\ell, x^\circ \vdash \omega} d(x) : D(x, x, 1_x)\end{array}}{\Gamma, x : A, y : A, z : \mathsf{hom}_A(x, y) \vdash_{\ell, x^\circ, y, z \vdash \omega} j_d^L(x, y, z) : D(x, y, z)}$$

One of our goals was to not divert too much from the spirit of MLTT: erasing the orientation will give us exactly the rules for the identity types of MLTT.

Similarly, we also have identity types, with the difference being that they are limited to isovariant terms.

Results

Internal 1-category theory. Working inside this theory, we are able to develop the theory of 1-categories in a synthetic manner, similar to how Homotopy Type Theory (HoTT) develops the theory of ∞-groupoids. For example, we derive:

Theorem (Yoneda)**.** *For A a type, a an element of A and P a presheaf, we have an equivalence*

$$\Pi_{x:A^\circ}(\mathsf{hom}_A(x, a) \to P(x)) \simeq P(a),$$

where $x : A^\circ$ indicates x appears isovariantly in $\mathsf{hom}_A(x, a) \to P(x)$.

1-Categorical semantics. We develop a semantic model of this type theory in the category of 1-categories. In this interpretation, the orientations described earlier are interpreted as endofunctors of Cat: $+$ is the identity, $-$ maps a category to its opposite category, and $\circ$ is the functor taking a category to its core (i.e. its maximal sub-groupoid).

Contexts $\Gamma \vdash_\ell$ are modeled as categories Γ^ℓ, while we model types A in a context $\Gamma \vdash_\ell$ as a functor from Γ^ℓ to Cat.

Future works

Following the developments in HoTT, we would also like to introduce an adequate notion of higher inductive types. Additionally, we also expect some version of directed univalence to be compatible with this theory. Finally, an important question is whether this theory can be extended to higher dimensions. We would like to investigate under what conditions the n-th iterated hom type gives precisely the n-cells of a higher category.

References

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