# A directed homotopy type theory for 1-categories

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# Motivation

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| Extensional Type Theory | $\rightsquigarrow$ | Sets |
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| | $\vdots$ | |
| **Directed Homotopy Type Theory** | $\rightsquigarrow$ | Categories |

# Some criteria for a directed HoTT

## Syntax

Extend HoTT:

- There is a type $\hom_A(a, b)$.
- We can do transport along morphisms in it.
- Be strong enough to prove classic theorems internally, e.g. the Yoneda lemma.

## Semantics

Extend the groupoid model:

- Types in the empty context are categories.
- The hom type is the hom-set.
- Functions $A \to B$ are functors $A \to B$.

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Our theory satisfies these!

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# The idea

1. We start with MLTT.
2. Add a hom type constructor, as in North [1].
3. Annotate the variances, $x \overset{\omega}{:} X$ with $\omega \in \{+, -, \circ\}$, as in Nuyts [2].
4. Import the rules we see in the semantics back to the syntax.

# The hom-type rules

$$\frac{\Gamma \vdash A : \mathcal{U}_k \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash \mathrm{hom}_A(a, b) : \mathcal{U}_k} \text{ hom-}\textsc{form}$$

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$$\frac{\Gamma \vdash A : \mathcal{U}_k \qquad \Gamma \vdash a : A}{\Gamma \vdash 1_a : \mathrm{hom}_A(a, a)}\ \text{hom-}\textsc{intro}$$

# The hom-type rules

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$$\frac{\Gamma^\ell \vdash A \overset{+}{:} \mathcal{U}_k \qquad \Gamma^\ell \vdash a \overset{\circ}{:} A}{\Gamma^\ell \vdash 1_a \overset{\circ}{:} \mathrm{hom}_A(a, a)} \text{ hom-INTRO}$$

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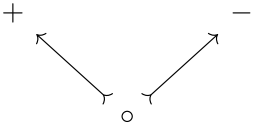
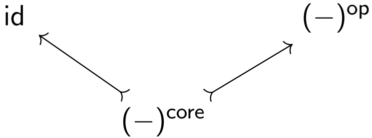
$$\frac{\begin{array}{c}\Gamma^\ell, x \overset{\circ}{:} A, y \overset{+}{:} A, \varphi \overset{k}{:} \mathrm{hom}_A(x,y) \vdash D(x,y,\varphi) \overset{+}{:} \mathcal{U}_k \\ \Gamma^\ell, x \overset{\circ}{:} A \vdash d(x) \overset{\omega}{:} D(x,x,1_x)\end{array}}{\Gamma^\ell, x \overset{\circ}{:} A, y \overset{+}{:} A, \varphi \overset{k}{:} \mathrm{hom}_A(x,y) \vdash j_d(x,y,\varphi) \overset{\omega}{:} D(x,y,\varphi)} \text{ ELIM-L}$$

# Orientations

We interpret the orientations $\{+, -, \circ\}$ as endofunctors $\mathsf{Cat} \to \mathsf{Cat}$.

They have an induced order:




And an induced multiplication:

| $\cdot$ | $\circ$ | $-$ | $+$ |
|---|---|---|---|
| $\circ$ | $\circ$ | $\circ$ | $\circ$ |
| $-$ | $\circ$ | $+$ | $-$ |
| $+$ | $\circ$ | $-$ | $+$ |

# The interpretation

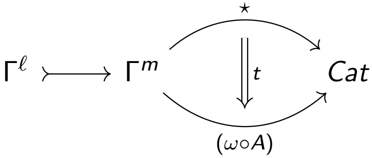
- Contexts $\rightsquigarrow$ Categories
- Empty context $\rightsquigarrow \star$
- Types in context $\rightsquigarrow$ Functors
  - $\left(\Gamma^\ell \vdash A \overset{\circ}{:} \mathcal{U}\right) \rightsquigarrow \left(\Gamma^\ell \to \mathrm{Cat}^{\mathrm{core}}\right)$
  - $\left(\Gamma^\ell \vdash A \overset{+}{:} \mathcal{U}\right) \rightsquigarrow \left(\Gamma^\ell \to \mathrm{Cat}\right)$
  - $\left(\Gamma^\ell \vdash A \overset{-}{:} \mathcal{U}\right) \rightsquigarrow \left(\Gamma^\ell \to \mathrm{Cat}^{\mathrm{op}}\right)$
- $\left(\Gamma^\ell, x \overset{\omega}{:} A\right) \rightsquigarrow \left(\Gamma^\ell.(\omega \circ A)\right)$
- $\left(\Gamma^\ell \vdash x \overset{\omega}{:} A\right) \rightsquigarrow$
  Sections $\left(\Gamma^\ell \to \Gamma^\ell.(\omega \circ A)\right)$
  Equivalently, $\star \implies (\omega \circ A)$

Hence, a closed type $\cdot \vdash A \overset{+}{:} \mathcal{U}$ is interpreted as a category $A$, and a judgement $a \overset{+}{:} A \vdash b \overset{+}{:} B$ is interpreted as a functor $A \to B$.

# Weakening

We can get more structure from the orientations, we can *weaken*:

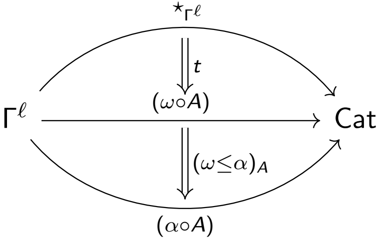
$$\frac{\ell \leq m \qquad \Gamma^m \vdash t \overset{\omega}{:} A}{\Gamma^\ell \vdash t \overset{\omega}{:} A}\ \textsc{Ort-weak-L}$$



# Weakening

Similarly, we can weaken on the right by postcomposing:

$$\frac{\omega \leq \alpha \qquad \Gamma^{\ell} \vdash t \overset{\omega}{:} A}{\Gamma^{\ell} \vdash t \overset{\alpha}{:} A} \text{ ORT-WEAK-R}$$



# Transmutation

We can switch all orientations.

$$\frac{\Gamma^{\ell} \vdash x \overset{m}{:} X}{\Gamma^{\ell \cdot \omega} \vdash x \overset{m \cdot \omega}{:} X} \textsc{Transmut}$$

In particular, we can make all annotations $\circ$.

# The hom constructor

We recall the rules

$$\frac{\Gamma^\ell \vdash A \overset{+}{:} \mathcal{U}_k \qquad \Gamma^\ell \vdash a \overset{-}{:} A \qquad \Gamma^\ell \vdash b \overset{+}{:} A}{\Gamma^\ell \vdash \hom_A(a, b) \overset{+}{:} \mathcal{U}_k} \text{ hom-FORM}$$

$$\frac{\Gamma^\ell \vdash A \overset{+}{:} \mathcal{U}_k \qquad \Gamma^\ell \vdash a \overset{\circ}{:} A}{\Gamma^\ell \vdash 1_a \overset{\circ}{:} \hom_A(a, a)} \text{ hom-INTRO}$$

We now interpret

$$a \overset{-}{:} A, b \overset{+}{:} A \vdash \hom_A(a, b) \overset{+}{:} \mathcal{U}_k \quad \rightsquigarrow \quad \hom : A^{\mathrm{op}} \times A \to \mathit{Cat}$$

$$a \overset{\circ}{:} A \vdash 1_a \overset{\circ}{:} \hom_A(a, a) \quad \rightsquigarrow \quad 1_a : A^{\mathrm{core}} \to \hom(a, a)^{\mathrm{core}}$$

# The hom constructor

We recall the rules

$$\frac{\Gamma^\ell \vdash A \overset{+}{:} \mathcal{U}_k \qquad \Gamma^\ell \vdash a \overset{-}{:} A \qquad \Gamma^\ell \vdash b \overset{+}{:} A}{\Gamma^\ell \vdash \mathrm{hom}_A(a, b) \overset{+}{:} \mathcal{U}_k} \text{ hom-FORM}$$

$$\frac{\Gamma^\ell \vdash A \overset{+}{:} \mathcal{U}_k \qquad \Gamma^\ell \vdash a \overset{\circ}{:} A}{\Gamma^\ell \vdash 1_a \overset{\circ}{:} \mathrm{hom}_A(a, a)} \text{ hom-INTRO}$$

We now interpret

$$a \overset{-}{:} A, b \overset{+}{:} A \vdash \mathrm{hom}_A(a, b) \overset{+}{:} \mathcal{U}_k \quad \rightsquigarrow \quad \mathrm{hom} : A^{\mathrm{op}} \times A \to Cat$$

$$a \overset{\circ}{:} A \vdash 1_a \overset{\circ}{:} \mathrm{hom}_A(a, a) \quad \rightsquigarrow \quad 1_a : A^{\mathrm{core}} \to \mathrm{hom}(a, a)$$

# And some other types

We also have some other type constructors

- Sigma types
- Pi types
- An op type constructor
- A core type constructor
- Identity types

Their rules are adapted to take in account variance.

# Example: Product types

There are now 9 product types:

$$\frac{\Gamma^{\ell} \vdash A \overset{+}{:} \mathcal{U}_k \qquad \Gamma^{\ell} \vdash B \overset{+}{:} \mathcal{U}_k}{\Gamma^{\ell} \vdash A^{\omega_1} \times B^{\omega_2} \overset{+}{:} \mathcal{U}_k} \times\text{-FORM}$$

$$\frac{\Gamma^{\ell} \vdash a \overset{\omega_1}{:} A \qquad \Gamma^{\ell} \vdash b \overset{\omega_2}{:} B}{\Gamma^{\ell} \vdash (a, b) \overset{+}{:} A^{\omega_1} \times B^{\omega_2}} \times\text{-INTRO}$$

# Expressivity

- Using these, and considering isovariant contexts $\Gamma^\circ$, we can develop the usual theory of HoTT.
- In particular, we have homotopies and equivalences as usual.
- We can reason internally about the hom-types, e.g.:
  - Reflexivity, composition, but not symmetry.
  - Functoriality.
  - Yoneda[1].

[1] With some caveats.

# Summary

- We annotate variances in variables $x \overset{\omega}{:} X$.
- We internalize rules that we see in the semantics:
  - We add structural rules concerning this variance.
  - All other rules are adjusted in terms of this variance.
- From these rules, we can use HoTT as usual as well as reason about 1-categories.

# Future work

- What's the relationship between isomorphisms and equality?
- Are $(\infty, 1)$-categories a model?
- What Directed Higher Inductive Types can be added?
- How much category theory can be proved internally?

*Thank you!*

# References


[1] Paige Randall North. "Towards a directed homotopy type theory". In: *Electronic Notes in Theoretical Computer Science* 347 (2019), pp. 223–239.

[2] Andreas Nuyts. "Towards a directed homotopy type theory based on 4 kinds of variance". In: *Mém. de mast. Katholieke Universiteit Leuven* (2015).